

A note on Diophantine approximation in $\mathrm{SL}_2(\mathbb{R})$.

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1. Two-dimensional lattices and 2×2 matrices.

Suppose that $\rho > 0$. For reals A_1, B_1 we define

$$A_2 = -\rho A_1, B_2 = -\rho B_1. \quad (1)$$

Let $r \in (0, 1)$ be a positive constant such that for any $A_1, B_1 \asymp Q$ the equation

$$xy - zw = 1 \quad (2)$$

has a solution $x, y, z, w \in \mathbb{Z}$ with

$$A_1 - Q^r \leq x \leq A_1 + Q^r, B_1 - Q^r \leq z \leq B_1 + Q^r, A_2 - Q^r \leq y \leq A_2 + Q^r, B_2 - Q^r \leq w \leq B_2 + Q^r. \quad (3)$$

Of course r does not depend on ρ . (We can take as r any number greater than $3/4$, see Section 5).

Theorem 1. *Suppose that we have two unimodular matrices*

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}). \quad (4)$$

Suppose that $\delta \neq 0$ and $\beta/\delta \notin \mathbb{Q}$. Then there exists a sequence of reals $t_\nu \rightarrow \infty, \nu \rightarrow \infty$ and a sequence of unimodular integer matrices

$$\begin{pmatrix} l_1^1 & l_1^2 \\ l_2^1 & l_2^2 \end{pmatrix} = \begin{pmatrix} l_1^1(\nu) & l_1^2(\nu) \\ l_2^1(\nu) & l_2^2(\nu) \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

such that

$$\begin{pmatrix} 1 & t_\nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} l_1^1 & l_1^2 \\ l_2^1 & l_2^2 \end{pmatrix} - \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} = O(|t_\nu|^{\frac{r-1}{r+1}}), \quad (5)$$

where the constant in $O(\cdot)$ may depend on the size of matrices (4).

Corollary. *Let $\underline{\Lambda}, \Gamma$ be two unimodular lattices in \mathbb{R}^2 . Suppose that*

$$\underline{\Lambda} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mathbb{Z}^2.$$

and $\beta/\delta \notin \mathbb{Q}$. Then there exists a sequence of reals $t_\nu \rightarrow \infty, \nu \rightarrow \infty$ such that for the lattice $\underline{\Lambda}_{t_\nu} = \underline{\Lambda} \cdot \begin{pmatrix} 1 & 0 \\ t_\nu & 1 \end{pmatrix}$ from the orbit of the lattice $\underline{\Lambda}$ in the horocycle flow on $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R})$ one has

$$\mathrm{dist}(\underline{\Lambda}_{t_\nu}, \Gamma) = O(|t_\nu|^{\frac{r-1}{r+1}})$$

(here we consider the natural distance in the space of lattices (see [1])).

Theorem 2. Consider a function $\psi(t)$ decreasing to zero as $t \rightarrow +\infty$ and such that $\psi(t) = O(t^{-1})$. Define $\rho(t)$ to be the function inverse to the function $t \mapsto 1/\psi(t)$. Suppose that under the conditions of Theorem 1 one has

$$\min_{p \in \mathbb{Z}} |(\beta/\delta) \cdot q - p| \geq \psi(q), \quad \forall q = 1, 2, 3, \dots \quad (6)$$

Then there exists a positive constant C such that for any $T \geq 1$ there exists a solution

$$\begin{pmatrix} l_1^1 & l_1^2 \\ l_2^1 & l_2^2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \quad t \in \mathbb{R}$$

of the system

$$\left\| \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} l_1^1 & l_1^2 \\ l_2^1 & l_2^2 \end{pmatrix} - \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} \right\| \leq CT^{\frac{r}{1+r}} / \rho \left(T^{\frac{1}{1+r}} \right), \quad 1 \leq |t| \leq T. \quad (7)$$

Here $\|\cdot\|$ stands for the maximum of absolute values of elements of a matrix.

Example. If $\psi(t) = t^{-\omega}$, $\omega \geq 1$ then $\rho(t) = t^{1/\omega}$ and the right hand side of (7) is equal to

$$O \left(T^{\frac{r}{1+r} - \frac{1}{\omega(1+r)}} \right).$$

This bound is non-trivial provided $\omega < r^{-1}$. Moreover in the case when β/δ is a badly approximable number (that is, $\psi(t) = \kappa t^{-1}$ with a positive κ) the right hand side of (7) is equal to $T^{\frac{r-1}{r+1}}$; in this case we see that the result of Theorem 1 holds uniformly.

Note that when r is close to $3/4$ then the exponent $\frac{r-1}{r+1}$ is close to $-1/7$.

In the next section we discuss some important history. We give all the proofs in Sections 3–5.

2. Some history.

Results similar to our Theorem 1 are known for a long time. They are related to quantitative version of the famous Ratner's orbit closure theorem, in the simplest case of unipotent flow on $\mathrm{SL}_2(\mathbb{R})$. We refer to a wonderful book [4] and the bibliography therein, as well as to the ordinal paper [6] by M. Ratner. The qualitative result of such a type for the simplest case of $\mathrm{SL}_2(\mathbb{R})$ was known much earlier.

Certain results similar to our Theorem 1 are due to A. Strömbergsson [8] and F. Maucourant and B. Weiss [3]. From their theorems a result of the same form as our Theorem 1 follows immediately, but in the right hand side of (5) then we have $O(t_\nu^{-\delta})$ with an effective positive δ which is not calculated explicitly. We do not compare our exponent to those from [8, 3].

However papers [8, 3] rely on the methods of dynamical systems. In the present paper we use different approach. We work with elementary theory of continued fractions and apply A. Weil's bounds for Kloostermann sums. Our continued fractions' consideration is connected with a paper by M. Laurent and A. Nogueira [2].

3. Lemmata.

We consider Euclidean plane \mathbb{R}^2 with coordinates (u, v) . Suppose that $|\alpha\delta - \beta\gamma| = 1$ and put

$$M = 4 \max(|\alpha|, |\beta|, |\gamma|, |\delta|, |\delta|^{-1})$$

Consider the lattice

$$\Lambda = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \mathbb{Z}^2.$$

The following lemma is a simple result from continued fractions' theory.

Lemma 1. *Suppose that $\beta/\delta \notin \mathbb{Q}$. Let p_ν/q_ν and $p_{\nu+1}/q_{\nu+1}$ be two consecutive convergent fractions to β/δ . Then*

- (i) $\begin{vmatrix} q_\nu & q_{\nu+1} \\ p_\nu & p_{\nu+1} \end{vmatrix} = (-1)^\nu;$
- (ii) *vectors*

$$\mathbf{e}_1 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} q_\nu - \begin{pmatrix} \gamma \\ \delta \end{pmatrix} p_\nu, \quad \mathbf{e}_2 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} q_{\nu+1} - \begin{pmatrix} \gamma \\ \delta \end{pmatrix} p_{\nu+1}$$

form a basis of Λ ;

- (iii) *the set*

$$\Pi_\nu = \{(u, v) \in \mathbb{R}^2 : |u| \leq Mq_{\nu+1}, |v| \leq Mq_{\nu+1}^{-1}\} \subset \mathbb{R}^2$$

contains a fundamental domain with respect to Λ , and hence any its shift $\mathbf{e} + \Pi_\nu$, $\mathbf{e} \in \mathbb{R}^2$ contains at least one point of Λ .

Corollary 1. *For any η and $R \in \mathbb{Z}_+$ the set*

$$\Pi_\nu(\eta; R) = \{(u, v) \in \mathbb{R}^2 : |u| \leq 2MRq_{\nu+1}, |v - \eta| \leq 2MRq_{\nu+1}^{-1}\}$$

contains $(2R+1)^2$ points of the lattice Λ of the form

$$\mathbf{e}_0 + l_1 \mathbf{e}_1 + l_2 \mathbf{e}_2, \quad l_j \in \mathbb{Z}, \quad |l_j| \leq R, \quad (8)$$

with some $\mathbf{e}_0 \in \Lambda$.

Lemma 2. *There exist $\mathbf{f}_1, \mathbf{f}_2 \in \Lambda$ such that*

- (i) $\mathbf{f}_1, \mathbf{f}_2$ *form a basis of Λ ;*
- (ii) $\mathbf{f}_j \in \Pi_\nu(\eta_j, q_{\nu+1}^r)$, $j = 1, 2$.

Proof. Consider lattice points of the form (8) from Corollary 1 for parameters η_1 and η_2 . Let these lattice points be

$$\mathbf{e}_0^1 + l_1^1 \mathbf{e}_1 + l_2^1 \mathbf{e}_2 \quad (9)$$

for parameter η_1 and

$$\mathbf{e}_0^2 + l_1^2 \mathbf{e}_1 + l_2^2 \mathbf{e}_2 \quad (10)$$

for parameter η_2 . We may suppose that $\eta_1 \neq 0$. Put $\rho = |\eta_2/\eta_1|$, $Q = q_{\nu+1}$, $R = Q^r = q_{\nu+1}^r$ and

$$\begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix} = \begin{pmatrix} q_\nu & q_{\nu+1} \\ -p_\nu & -p_{\nu+1} \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \eta_1 & \eta_2 \end{pmatrix}.$$

Then (1) is valid and $A_1 \asymp Q$. Under the multiplication by the matrix $\begin{pmatrix} q_\nu & q_{\nu+1} \\ -p_\nu & -p_{\nu+1} \end{pmatrix}^{-1}$ lattice points (9) and (10) turn into integer points $\begin{pmatrix} x \\ z \end{pmatrix}$ and $\begin{pmatrix} y \\ w \end{pmatrix}$ respectively, satisfying

(3). By the definition of r there exist x, y, z, w satisfying (2). So there exist l_i^j , $i, j = 1, 2$ such that two points (9) and (10) form a basis of Λ . \square

Lemma 3. *Suppose that*

$$\begin{vmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{vmatrix} = \begin{vmatrix} \Xi_1 & \Xi_2 \\ H_1 & H_2 \end{vmatrix} = 1. \quad (11)$$

Suppose that

$$\max_{j=1,2} |H_j - \eta_j| \leq \varepsilon \quad (12)$$

and $H_1, H_2 \neq 0$. Then

$$\left| \frac{\Xi_1 - \xi_1}{H_1} - \frac{\Xi_2 - \xi_2}{H_2} \right| \leq \frac{\varepsilon \cdot (|\xi_1| + |\xi_2|)}{|H_1 H_2|}. \quad (13)$$

Proof. As

$$\xi_1 \eta_2 - \Xi_1 H_2 - (\xi_2 \eta_1 - \Xi_2 H_1) = 0,$$

from (11) we see that

$$|(\xi_1 - \Xi_1)H_2 - (\xi_2 - \Xi_2)H_1| \leq \varepsilon \cdot (|\xi_1| + |\xi_2|).$$

So (13) follows. \square

4. Proof of theorems.

Proof of Theorem 1.

We take large ν . Let vectors \mathbf{f}_j from Lemma 2 be of the form $\mathbf{f}_j = \begin{pmatrix} \Xi_j \\ H_j \end{pmatrix}$, $j = 1, 2$. Then $H_1, H_2 \neq 0$ and $\max_{j=1,2} |H_j - \eta_j| \leq 2Mq_{\nu+1}^{r-1}$ and $\max_{j=1,2} |\Xi_j| \leq 2Mq_{\nu+1}^{1+r}$. Put $t_\nu = \frac{\xi_1 - \Xi_1}{H_1}$. Then by (13) we have

$$\left| t_\nu - \frac{\xi_2 - \Xi_2}{H_2} \right| = O(q_{\nu+1}^{r-1}).$$

Of course we have $|t_\nu| = O(q_{\nu+1}^{1+r})$. Now

$$\begin{pmatrix} 1 & t_\nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} l_1^1 & l_1^2 \\ l_2^1 & l_2^2 \end{pmatrix} = \begin{pmatrix} 1 & t_\nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Xi_1 & \Xi_2 \\ H_1 & H_2 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & t_\nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Xi_1 & \Xi_2 \\ H_1 & H_2 \end{pmatrix} - \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} = O(q_{\nu+1}^{r-1})$$

So Theorem 1 follows. \square

Proof of Theorem 2.

Given real $U \geq 1$ put $U_* = \rho(U)$. By Minkowski's convex body theorem and condition (6) one can take primitive point $(q, p) \in \mathbb{Z}^2$ such that

$$U_* \leq q \leq U. \quad |(\beta/\delta) \cdot q - p| \leq U^{-1}. \quad (14)$$

Then this point may be completed to a basis of \mathbb{Z}^2 by a point $(q', p') \in \mathbb{Z}^2$ such that

$$U_* \leq q' \leq 2U. \quad |(\beta/\delta) \cdot q - p| \leq U_*^{-1}. \quad (15)$$

From (14,15) we see that the rectangle

$$\{(u, v) \in \mathbb{R}^2 : |u| \leq MU, |v| \leq MU_*^{-1}\} \subset \mathbb{R}^2$$

contains a fundamental domain for Λ . Now we follow the argument of the proof of Theorem 1. We see that t may be taken to be $\ll U^{1+r}$, and we establish the bound

$$\left\| \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} l_1^1 & l_1^2 \\ l_2^1 & l_2^2 \end{pmatrix} - \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} \right\| \ll U^r U_*^{-1}.$$

By putting $U = T^{\frac{1}{1+r}}$ we have

$$U^r U_*^{-1} = T^{\frac{r}{1+r}} / \rho \left(T^{\frac{1}{1+r}} \right),$$

and Theorem 2 follows. \square

5. About admissible value of r .

Here we show that any value $r > 3/4$ is good for our purpose. First of all we may suppose that $w = p$ is a prime number (here we use a well-known fact that between Q and $Q + Q^{3/4}$ for large Q there exists a prime number, see [5]).

Then we apply the well-known fact that for any two intervals I_1, I_2 of lengths $\gg p^{3/4+\varepsilon}$ there exist $x \in I_1, y \in I_2$ such that $xy \equiv 1 \pmod{p}$ (see [7, 9]). This result follows from A. Weil's bounds for Kloostermann sums. Now $z = \frac{xy-1}{p}$ will be an integer. Easy calculation shows that x, y, z, w will satisfy (3).

References

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